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On The Extension of Cauchy-Goursat Theorem to Multiply Connected Region and Using It to Compute Complicated Real Integrals

[1]Ezra E. T., [2]Manjak N. H., [3]Kwami A. M., [4]Okai J. O.
[1,2,3,4] *Department of Mathematical Sciences, Abubakar Tafawa Balewa University, Bauchi, Nigeria*

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ABSTRACT
Many authors have presented proves for the Cauchy-Goursat theorem on a simply connected domain outside the conventional way of proving it. In this research, we have studied and presented two proves using different approach to prove the Cauchy-Goursat theorem on a multiply connected domain. The first approach was achieved by converting the multiply connected domain to a simply connected domain and from the second approach we isolated the point at which the function is not analytic and provided a condition that the function satisfy the Cauchy's theorem. We also provides some examples to demonstrate how the theorem is applied to solve both real and complex valued integral problems.

Keywords: Cauchy-Goursat, Analytic function, Simply connected region, Multiply connected region and Continuity of function.

I. INTRODUCTION

Complex analysis is an extension of real calculus which brings out hidden properties in the real calculus (Azram et al 2010 and Erwin et al2011). In the study of complex analysis, complex integration is a very important and significant concept and which also serves as an elegant, powerful and a useful tool for mathematicians, physicist and engineers (Azram et al 2013). However, complex analysis has a deep connection with other fields of mathematics and physics, such as algebraic geometry, quantum mechanics, number theory (Murray et al2009)).

Cauchy-Goursat theorem is an important fundamental and well celebrated result of the complex integral calculus which is frequently applied in conformal mappings, electrical engineering, quantum mechanics, mathematical physics e.t.c. (Alpay, D. 2015 and Ahlfors, L. V. 1979). The theorem serves as a basis for the Cauchy integral formula which has many applications in various areas of mathematics, having a long history in complex analysis, combinatorics, discrete mathematics, or number theory(Brownet al2009 and Conway, J. B 1986). Recently, it has been used to derive an exact integral formula for the coefficients of cyclotomic and other classes of polynomials and Poisson integral formula (Leslie, C., 2014). Also, the theorem focus on analytic functions which is a very important concept in the study of complex variables of which all those functions are endowed with a very strong inner structure (Churchill, R. V. and James W. B., 2003).

The theorem was first established by Augustin-Louis Cauchy between 1789-1857 and it was later extended by Edouard Goursat between 1858-1936, of which the theorem was named after them as Cauchy-Goursat theorem or simply Cauchy's theorem (Scott, A. E. 1978 and McCharty, C. A. 1975). To clearly understand the Cauchy-Goursat theorem of complex integration, one need to acquaint himself with the concept of differentiability, analytic functions and regions in the complex plane (simply or multiply connected region) upon which the function is been define (Williams, A.R. 2022).

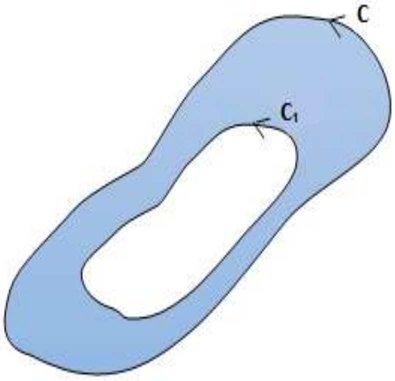
II. CAUCHY-GOURSAT THEOREM FOR MULTIPLY CONNECTED DOMAIN

Theorem 2.1
Let C be a simple closed curve such that C_1 is a simple closed curve lying in C. If f(z) is analytic on C, C_1 and the region bounded by the two curves then

$$\oint_c f(z)dz = \oint_{c_1} f(z)dz \quad .....(1)$$

Proof:

Construct a simple closed curve C which encloses another simple closed curve C_1 positively oriented as in figure1. We will connect the two circles by cross cutting them and joining a point Mon C to a point Q on C_1 as shown in figure 2.



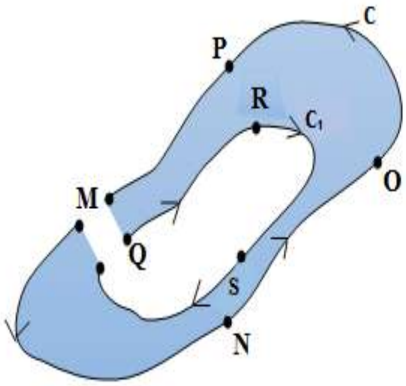


Figure1: Multiply connected Domain Figure 2: A Cross Cut Domain

Now, observe that MNOPMQRSQM is a simply connected domain. Clearly, f(z) is analytic in this domain and is continuous on its boundary. Hence by Cauchy's theorem for simply connected domain, we get

$$\oint_{MNOPMQRSQM} f(z)dz = 0$$

This can be integrated part by part to obtain

$$\oint_{MNOPMQRSQM} f(z)dz = \oint_{MNOPM} f(z)dz + \oint_{MQ} f(z)dz + \oint_{QRSQ} f(z)dz + \oint_{QM} f(z)dz \quad \text{…..(2)}$$

$$\Rightarrow \oint_{MNOPM} f(z)dz + \oint_{MQ} f(z)dz + \oint_{QRSQ} f(z)dz + \oint_{QM} f(z)dz = 0 \quad \text{…..(3)}$$

Since the integral over MQ and QM is the same but opposite in direction, therefore cancelled out.

$$\oint_{MNOPM} f(z)dz + \oint_{MQ} f(z)dz + \oint_{QRSQ} f(z)dz - \oint_{QM} f(z)dz = 0$$

(Since $\oint_{MQ} f(z)dz = -\oint_{QM} f(z)dz$)

$$\oint_{MNOPM} f(z)dz + \oint_{QRSQ} f(z)dz = 0 \quad \text{…..(4)}$$

$$\oint_{c} f(z)dz + \oint_{-c_1} f(z)dz = 0$$ (Where $c = MNOPM$ and $-c_1 = QRSQ$)

$$\oint_{c} f(z)dz - \oint_{c_1} f(z)dz = 0 \quad \text{…..(5)}$$

$$\oint_{c} f(z)dz = \oint_{c_1} f(z)dz$$ As required.

Theorem 2.2

If a function f(z) is analytic at all points inside and on a simply connected region bounded by C except at a point z_0 inside C. Then $\oint_c f(z)dz = \oint_{c_1} f(z)dz$ and $\oint_c f(z)dz = 0$ if $\lim_{z\to z_0} f(z) = 0$ where $c_1: |z - z_0| = r$ (Jiarui et al 2021).

Proof

Construct a positively oriented simple closed curve C_1 of radius r centered at z_0 as in the hypothesis that is $C_1: |z - z_0| = r$.

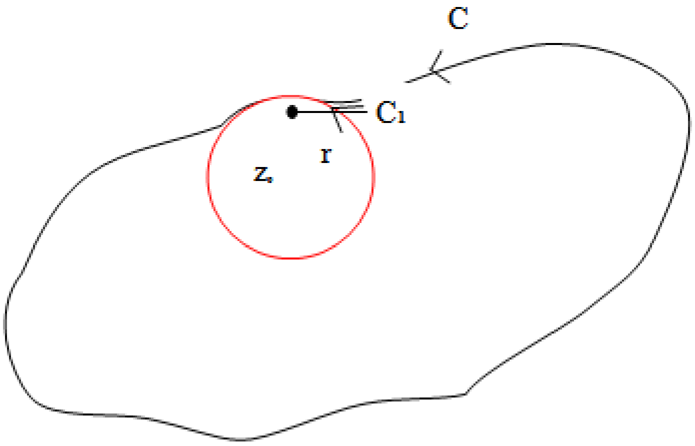


Figure 3: Positively Oriented Curve

Suppose $f(z)$ is analytic inside and on the closed curve C except at the point z_0. Since $lim_{z\to z_0} f(z) = 0$, then by definition implies given $\varepsilon>0$, $\exists\ \delta>0$ such that $f(z)-0<\varepsilon$ whenever $0<z-z0<\delta$.
Since $lim_{z\to z_0} f(z) = 0 \Rightarrow lim_{z\to z_0}(z - z_0)f(z) = 0$ given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|(z - z_0)f(z) - 0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

$$\Rightarrow |z - z_0||f(z)| < \varepsilon \Rightarrow |f(z)| < \frac{\varepsilon}{|z - z_0|} = \frac{\varepsilon}{r}$$

By the theorem 2.1

$$\Rightarrow \oint_c f(z)dz = \oint_{c_1} f(z)dz \qquad .....(6)$$

$$\Rightarrow \left|\oint_c f(z)dz\right| = \left|\oint_{c_1} f(z)dz\right| \le |f(z)|\left|\oint_{c_1} dz\right| < \frac{\varepsilon}{r}2\pi r \to 0 \text{ as } \varepsilon \to 0 \qquad .....(7)$$

$$\Rightarrow \left|\oint_c f(z)dz\right| = 0 \text{ as } \varepsilon \to 0 \qquad .....(8)$$

Hence, $\oint_c f(z)dz = 0$ as required.

Theorem 2.3

Let $f(z)$ be Meromophic on $\mathbb{C}$, having only a finite number of poles, not lying on the real axis. Suppose that there exist a constant K such that $|f(z)| \le K/|z|$ for all sufficiently large $|z|$. Let $a > 0$. Then

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \sum \operatorname{Re}s\left(e^{iaz} f(z)\right) \qquad .....(9)$$

in the upper half plane.

Proof

Suppose the complexified function $f(z)e^{iaz}$ has a finite number of singularities (poles), in the upper half plane. Then by Residue theorem

$$\int_C f(z)e^{iaz}dz = 2\pi i \sum \operatorname{Re}s(f(z)e^{iaz}) \qquad .....(10)$$

For all poles that lies in the upper half plane.

For the sake of understanding, take $a=1$. Then we integrate over any rectangle with vertices at $-a,b,$ $b+i\beta$ and $-a+i\beta$, where $\beta = a+b$ as shown in figure 4 below.

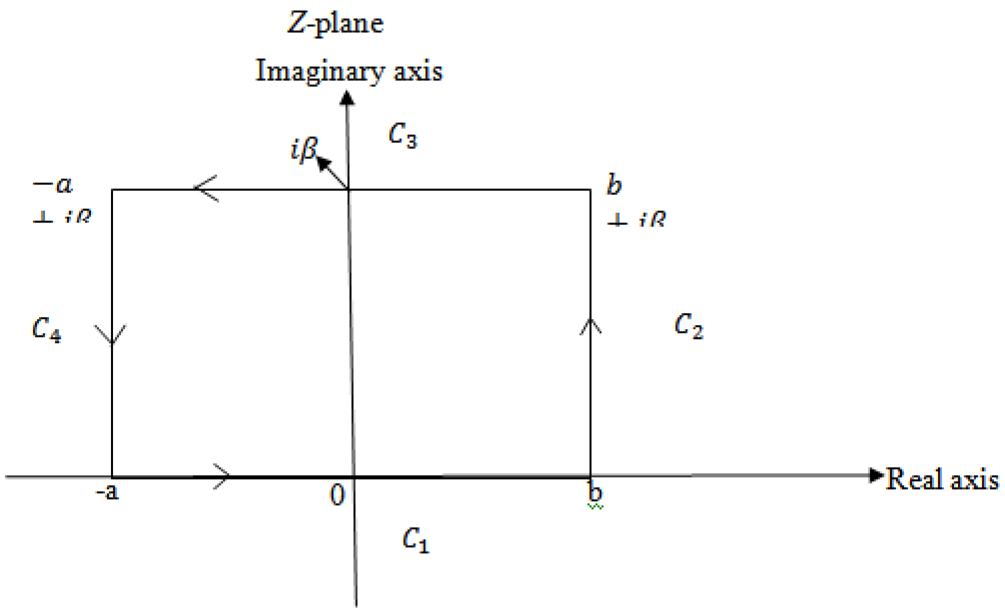


Figure 4: A rectangle on the complex plane

Take the curve $C = C_1 + C_2 + C_3 + C_4$ so that

$$\int_C f(z)e^{iz}dz = \int_{C_1+C_2+C_3+C_4} f(z)e^{iz}dz = \int_{C_1} f(z)e^{iz}dz + \int_{C_2} f(z)e^{iz}dz + \int_{C_3} f(z)e^{iz}dz + \int_{C_4} f(z)e^{iz}dz$$

....(11)

Taking $a,b>0$ sufficiently large, it suffices to prove that integral over c_2, c_3 and c_4 tend to 0 as $a,b \to \infty$.

First we note that $e^{iz} = e^{i(x+iy)} = e^{ix} \cdot e^{-y} = e^{-y}[\cos x + i \sin x]$ ....(12)

So that $\left|e^{iz}\right| = e^{-y} \to 0$ as $y \to \infty$

Case I: Consider the integral $\int_{C_2} f(z)e^{iz}dz$

Parameterizing z as $z = b+iy$, where $0 \le y \le \beta$ along C_2 so that

$dz = idy$ and $e^{iz} = e^{ib} \cdot e^{-y}$

....(13)

Hence $\int_{C_2} f(z)e^{iz}dz = i\int_0^{\beta} e^{ib}e^{-y} f(b+iy)dy$ from equation (12) and (13) so that

$$\left|\int_{C_2} e^{iz} f(z)dz\right| = \left|i\int_0^{\beta} e^{ib}e^{-y} f(b+iy)dy\right| \le |f(b+iy)|\left|e^{ib}e^{-y}\right|\left|\int_0^{\beta} dy\right| \le |f(b+iy)|\left|\int_0^{\beta} e^{-y}dy\right| < \tfrac{K}{b}(1-e^{-\beta})$$

i.e. $\left|\int_{C_2} e^{iz} f(z)dz\right| \le \frac{K}{b}(1-e^{-\beta}) \to 0$ and $b \to \infty$

$$\int_{C_2} e^{iz} f(z)dz = 0 \qquad \text{.....(14)}$$

Case II: We again consider the integral $\int_{C_3} e^{iz} f(z)dz$. Parameterizing the curve C_3, we have $z = x + i\beta$, $-a \le x \le b$ so that $dz = dx$. Therefore

$\int_{C_3} e^{iz} f(z)dz = \int_{-a}^{b} e^{ix}e^{-\beta} f(x+i\beta)dx$ and

$$\left|\int_{C_3} e^{iz} f(z)dz\right| = \left|\int_{-a}^{b} e^{ix}e^{-\beta} f(x+i\beta)dx\right| \le \left|e^{ix}e^{-\beta}\right|\left|f(x+i\beta)\right|\left|\int_{-a}^{b} dx\right| \le e^{-\beta}\tfrac{K}{\beta}(b+a) = e^{-\beta}K \to 0$$

As $\beta \to \infty\ a,b \to \infty$

$$\int_{C_3} e^{iz} f(z)dz = 0 \qquad \text{.....(15)}$$

Case III: We consider the integral along C_4 i.e $\int_{C_4} e^{iz} f(z)dz.$ Parameterizing as usual, we have $z = -a + iy,$ $0 \le y \le \beta$ along $C_4 \Rightarrow dz = idy$ and $e^{iz} = e^{-ia}e^{-y}$

Substituting into the integral we have

$$\int_{C_4} f(z)e^{iz}dz = i\int_0^{\beta} e^{-ia}e^{-y} f(-a+iy)dy$$

$$\left|\int_{C_4} f(z)e^{iz}dz\right| = \left|i\int_0^{\beta} e^{-ia}e^{-y} f(-a+iy)dy\right| \le \left|e^{-ia}e^{-y}\right|\left|f(-a+iy)\right|\left|\int_0^{\beta} dy\right| \le e^{-\beta}\tfrac{K}{\beta}\cdot\beta \le \tfrac{K}{e^{-y}} \to 0$$

as $a,b \to 0$

$$\int_{c_4} e^{iz} f(z)dz = 0 \qquad \text{.....(16)}$$

Case IV: Finally $\int_{C_1} f(z)e^{iz}dz.$ Parameterizing, we have along c_1, $z = x, dz = dx$ and $e^{iz} = e^{ix}, -a \le x \le b$

$\Rightarrow \int_{C_1} f(z)e^{iz}dz = \int_{-a}^{b} f(x)e^{ix}dx$ and taking limit.

$$\Rightarrow \int_{C_1} f(z)e^{iz}dz = \lim_{a,b\to\infty}\int_{-a}^{b} f(x)e^{ix}dx = \int_{-\infty}^{\infty} f(x)e^{ix}dx \qquad \text{.....(17)}$$

Therefore from equation (11) we have

$$\int_{c} f(z)e^{iz}dz = \int_{C_1} f(z)e^{iz}dz + \int_{C_2} f(z)e^{iz}dz + \int_{C_3} f(z)e^{iz}dz + \int_{C_4} f(z)e^{iz}dz$$

$$= \int_{-\infty}^{\infty} f(x)e^{ix}dx + 0 + 0 + 0 = \int_{-\infty}^{\infty} f(x)e^{ix}dx \qquad \text{.....(18)}$$

Now comparing equation (10) and (11) we see that

$$\int_{C} f(z)e^{iz}dz = \int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum \text{Re}\, s\left[f(z)e^{iz}\right]$$ as required.

Note that the summation is all over Residues of poles that lie in the upper half plane.

III. APPLICATION

Example 3.1:

Evaluate $\oint_{c} \frac{dz}{z-a}$ where C is a simple closed curve (Juan, C. P., 2016).

(a) Outside C and

(b) Inside C

Solution:

In this case, $f(z) = \frac{1}{z-a}$ which is analytic on C except $z = a$.

Construct a simple closed curve C which encloses a circle γ centered at a and radius α

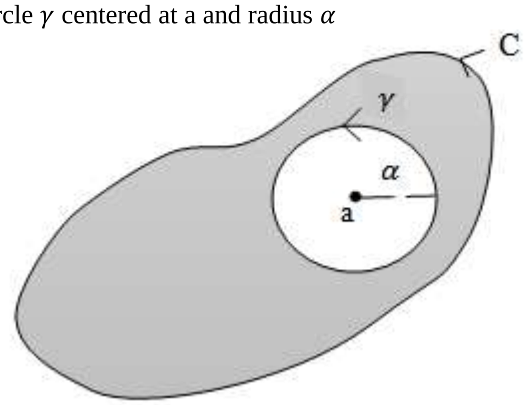


Figure 5: A simple closed curve C which encloses γ

(a) Suppose a is inside C and let γ be a circle of radius α with center at $z = a$ so that γ is inside C.

Now taking the limit of the function f(z) at $z = a$, we have

$$f(z) = \frac{1}{z-a},$$

.....(19)

$$\lim_{z\to a} f(z) = \lim_{z\to a} \frac{1}{z-a} = \frac{1}{a-a} = \frac{1}{0} = \infty \neq 0$$

Thus, the value of the integral around γ will be therefore not be zero.

By theorem 2.1 we see;

$$\oint_{c} \frac{dz}{z-a} = \oint_{\gamma} \frac{dz}{z-a}$$

.....(20)

Now on γ, $|z-a| = \varepsilon$, i.e. $z = a + \varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$. Thus, since $dz = i\varepsilon e^{i\theta}$, the right side of (15) becomes;

$$\int_{0}^{2\pi} \frac{i\varepsilon e^{i\theta}d\theta}{\varepsilon e^{i\theta}} = i\int_{0}^{2\pi} d\theta = 2\pi i \neq 0$$

.....(21)

(b) If a is outside C, then $f(z) = \frac{1}{z-a}$ is analytic everywhere inside and on C. Hence, by Cauchy's theorem, $\oint_{c} \frac{dz}{z-a} = 0$.

Example 3.2:

We test the limit condition.

If C is a simple closed curve enclosing $z = 0$ and $z = 1$, then find $\int_{c} \frac{1}{z(z-1)} dz$ (John, H. M., & Russell, W. H. 2011).

Solution:

Construct a simple closed curve C which encloses the points 0 and 1. Again we construct circle C_1 centered at 0 and circle C_2 centered at 1 which lies completely inside C and non-intersecting.

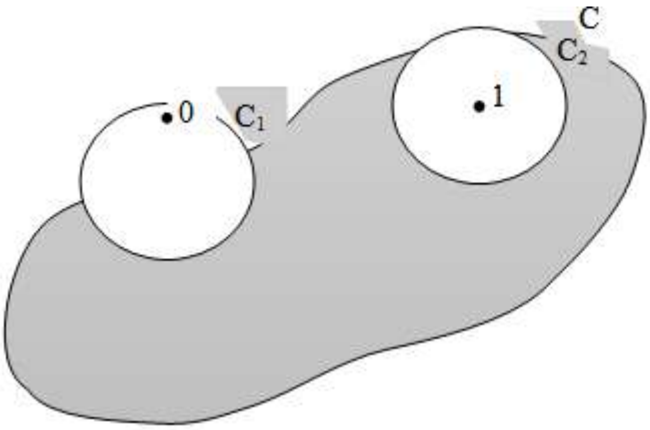


Figure 6: A simple closed curve containing C_1 and C_2.

We have $f(z) = \frac{1}{z(z-1)}$ .....(22)

By partial fraction we have

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z} \quad .....(23)$$

Now taking the limit of the function f(z) at $z = 0,1$

First at $z = 0$

$$\lim_{z\to 0}\left(\frac{1}{z-1} - \frac{1}{z}\right) = \left(\frac{1}{0-1} - \frac{1}{0}\right) = -1 - \infty = -\infty \neq 0 \quad .....(24)$$

Again $z = 1$,

$$\lim_{z\to 1}\left(\frac{1}{z-1} - \frac{1}{z}\right) = \left(\frac{1}{1-1} - \frac{1}{1}\right) = \infty - 1 = \infty \neq 0 \quad .....(25)$$

Thus, by theorem we have

$$\int_c \frac{1}{z(z-1)} dz = \int_{C_1} \frac{1}{z-1} dz - \int_{C_1} \frac{1}{z} dz + \int_{C_2} \frac{1}{z-1} dz - \int_{C_2} \frac{1}{z} dz \quad .....(26)$$

By Cauchy's theorem we obtain

$$\int_c \frac{1}{z(z-1)} dz = 0 - 2\pi i + 2\pi i - 0 = 0 \quad .....(27)$$

Example 3.3:

We show that for all complex number $\rho \in \mathbb{C}$ (Jiarui et al 2021)

$$e^{-\pi\rho^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\rho} dx \quad .....(28)$$

Solution:

Note that $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\rho} dx$ is the Fourier transform of $e^{-\pi\rho^2}$

For $\rho = 0$, we get from (28)

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1 \quad .....(29)$$

Which is obvious. But then using the fact that cosine and sine are even and odd functions respectively, we get from Euler's formula

i.e. $e^{i2\pi x\rho} = \cos(2\pi x\rho) + i\sin(2\pi x\rho)$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi x\rho} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} [\cos(2\pi x\rho) + i\sin(2\pi x\rho)] dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi x\rho) dx + i\int_{-\infty}^{\infty} e^{-\pi x^2} \sin(2\pi x\rho) dx \quad .....(30)$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi ix\rho} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi x\rho) dx \quad .....(31)$$

Which shows it is enough to prove for $\rho > 0$. Fixing $\rho > 0$, we integrate $f(z) = e^{-\pi z^2}$ along the boundary of the rectangle whose parameterizations (let $b = 2\pi\rho$) of the four sides of the rectangle are given as;

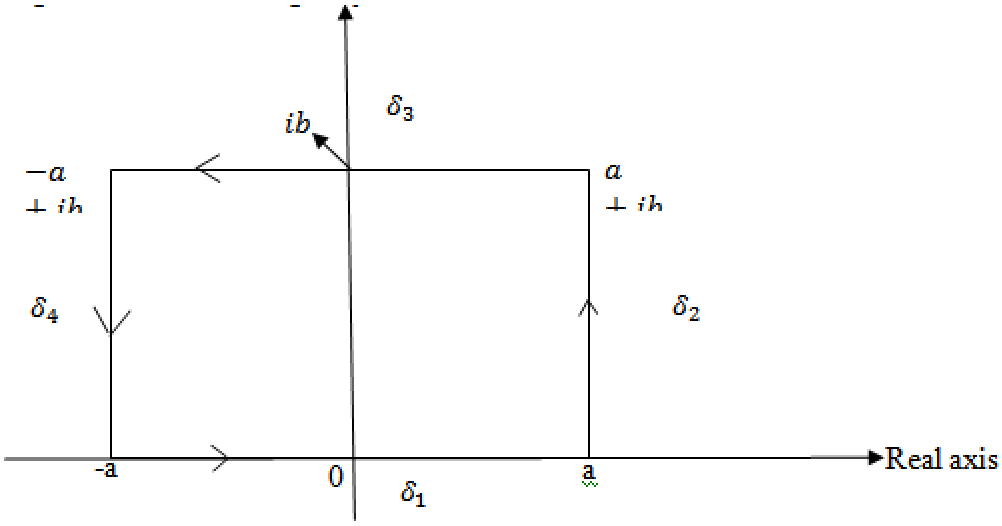


Figure 7: The geometrical parameterization of a rectangle

$\delta_1 = x, \quad x \in [-a, a]$
$\delta_2 = a + ix, \quad x \in [0, b] \quad .....(32)$
$\delta_3 = -x + ib, \quad x \in [-a, a]$
$\delta_4 = -a + i(b - x), \quad x \in [0, b]$

Since f is analytic in ℂ, then by Goursat's theorem, the integral is 0.
Now,

$$\int_{\delta_1} f(z)dz = \int_{-a}^{a} e^{-\pi x^2} dx \quad .....(33)$$

$$\int_{\delta_2} f(z)dz = \int_0^b e^{-\pi(a+ix)^2} dx = \int_0^b e^{-\pi(a^2+2iax-x^2)} dx = \int_0^b e^{-\pi a^2} e^{-2\pi iax} e^{\pi x^2} dx = ie^{-\pi a^2} \int_0^b e^{\pi x^2} e^{-iax} dx \quad .....(34)$$

$$\int_{\delta_3} f(z)dz = \int_{-a}^{a} e^{-\pi(-x+ib)^2} dx = \int_{-a}^{a} e^{-\pi(x^2-2ibx-b^2)} dx = \int_{-a}^{a} e^{-\pi a^2} e^{2\pi ibx} e^{\pi b^2} dx = -e^{-\pi b^2} \int_{-a}^{a} e^{-\pi x^2} e^{ibx} dx \quad ...(35)$$

$$\int_{\delta_4} f(z)dz = \int_0^b e^{-\pi[-a+i(b-x)]^2} dx = \int_0^b e^{-\pi[a^2-2ia(b-x)-(b-x)^2]} dx = \int_0^b e^{-\pi a^2} e^{-2\pi ia(b-x)} e^{\pi(b-x)^2} dx$$

$$= -ie^{-\pi a^2}\int_0^b e^{\pi(b-x)^2}e^{ia(b-x)}dx \qquad ....(36)$$

Hence, $\left|\int_{\delta_2} f\right|, \left|\int_{\delta_4} f\right| \le e^{-\pi a^2}e^{\pi b^2} \to 0$ for $a \to \infty$. Moreover,

$$\lim_{a\to 0}\int_{\delta_3} f = e^{-\pi b^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{ibx}dx, \text{ and } \lim_{a\to 0}\int_{\delta_1} f = \int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1 \qquad .....(37)$$

So letting $a \to \infty$ in $\sum_{n=1}^{4}\int_{\delta_n} f = 0$ that is

$$\int_{\delta_1} f + \int_{\delta_2} f + \int_{\delta_3} f + \int_{\delta_4} f = 0 \qquad .....(38)$$

$$1 + 0 + \left(-e^{\pi p^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi i x p}dx\right) + 0 = 0 \qquad .....(39)$$

$$1 - e^{\pi p^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi i x p}dx = 0 \qquad .....(40)$$

$$e^{\pi p^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi i x p}dx = 1 \qquad .....(41)$$

$$\Rightarrow e^{-\pi p^2} = \int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi i x p}dx \qquad .....(42)$$

IV. DISCUSSION OF RESULTS

Cauchy-Goursat theorem for multiply connected domain is studied and for the theorem to be applied to solve any complex integral problem, the function must be able to satisfy the condition of been analytic inside and on the boundary of a simple closed curve and if the function is having one or more singular point lying inside the contour, then the need to do away with those singular points will arise by either finding a way of making the domain (region) to be a simply connected domain or by constructing a small non-intersecting circle around each of the singular points and the singular points as the center of each circle. The prove of theorem (2.1) shows that the function is continuous which gives it a strong property i.e. the function is defined at all points on the boundary, the limit of the function also exists at each point on the boundary which gave us the idea for the prove of theorem (2.2) and suggest that the idea can be extend to some finite number of points. Singular point plays an important role in the study of complex integration. Example (3.1a) shows that when the singular point is inside the simple closed curve, then the region is a multiply connected region and applying the condition from theorem 2.2, we see that the limit is not zero showing that the integral along γ cannot be zero and (3.1b) shows that if a singular point is outside the simple closed curve, then the domain is just a simply connected domain and the Cauchy-Goursat theorem is applied at once to obtain the solution for the integral. Example (3.2) shows that the limit value of the function at points 0 and 1 is non-zero which shows that the value of the integral will not be zero but then by theorem the solution is zero, implying that the condition is only valid when the singular point is one. The method used in provingtheorem (2.3) was used in solving example (3.3).

V. CONCLUSION

Cauchy-Goursat theorem for both simply connected domain and multiply connected domain are very important in the study of complex integration. The proves we gave brought out the beauty of complex analysis and important roles played by limit value and continuity of functions and the connection between complex integrals and real integrals.

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